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Let ABC be a triangle with $BC = a, AC = b, AB = c$ and circumradius R. Show that

$$(b+c)/a^5 + (c+a)/b^5 + (a+b)/c^5 \geq 2/(3R^4).$$

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:Let F be are of $\triangle ABC$. Since $4F \cdot R = abc$ we have

$$\sum \frac{b+c}{a^5} \geq \frac{2}{3R^4} \Leftrightarrow \sum \frac{b^4c^4(b+c)}{a} \geq \frac{2 \cdot a^4b^4c^4}{3R^4} = \frac{512F^4}{3}.$$

Since by Cauchy Inequality $\sum \frac{b^4c^4(b+c)}{a} = \sum \frac{b^4c^4(b+c)^2}{a(b+c)} \geq \frac{\left(\sum b^2c^2(b+c)\right)^2}{2(ab+bc+ca)}$

and $3(ab+bc+ca) \leq (a+b+c)^2$ then $\sum \frac{b^4c^4(b+c)}{a} \geq \frac{3\left(\sum b^2c^2(b+c)\right)^2}{2(a+b+c)^2}$.

Thus, suffices to prove inequality

$$\frac{3\left(\sum b^2c^2(b+c)\right)^2}{2(a+b+c)^2} \geq \frac{512F^4}{3} \Leftrightarrow \frac{\sum b^2c^2(b+c)}{a+b+c} \geq \frac{32F^2}{3} \Leftrightarrow$$

(**1**) $\quad \sum b^2c^2(b+c) \geq \frac{64sF^2}{3}$.

But we can reduce inequality (**1**) to more simple inequality.

Indeed, since $x^2+y^2+z^2 \geq xy+yz+zx, \forall x,y,z \in \mathbb{R}$ then

$\sum b^2c^2(b+c) = (a+b+c)(a^2b^2+b^2c^2+c^2a^2) - abc(ab+bc+ca) = 0 \geq$
$(a+b+c) \cdot abc(a+b+c) - abc(ab+bc+ca) = abc(a^2+b^2+c^2+ab+bc+ca) \geq$
$2abc(ab+bc+ca)$.

Thus, remains to prove inequality $2abc(ab+bc+ca) \geq \frac{64sF^2}{3} \Leftrightarrow$

(**2**) $abc(ab+bc+ca) \geq \frac{32sF^2}{3}$.

Let $x := s-a, y := s-b, z := s-c, p := xy+yz+zx, q := xyz$. Then assuming $s = 1$ (due homogeneity of (**2**)) we obtain $x,y,z > 0, x+y+z = 1, a = 1-x, b = 1-y$, $c = 1-z, abc = p-q,\ ab+bc+ca = 1+p,\ F^2 = q$.

Since $3p = 3(y+yz+zx) \leq (x+y+z)^2 = 1$ and

$3q = 3xyz(x+y+z) \leq (y+yz+zx)^2 = p^2$ then

$abc(ab+bc+ca) - \frac{32sF^2}{3} = (p-q)(1+p) - \frac{32q}{3} \geq$

$\left(p - \frac{p^2}{3}\right)(1+p) - \frac{32 \cdot \frac{p^2}{3}}{3} = \frac{1}{9}p(p+9)(1-3p) \geq 0.$

Another proof of (2).

Since* $ab+bc+ca \geq 4\sqrt{3}F$ and $abc = 4RF$ then

$abc(ab+bc+ca) \geq \frac{32sF^2}{3} \Leftrightarrow 4RF \cdot 4\sqrt{3}F \geq \frac{32sF^2}{3} \Leftrightarrow R\sqrt{3} \geq \frac{2s}{3} \Leftrightarrow$

$a+b+c \leq 3\sqrt{3}R$ (well known geometric inequality).

* Follows from Hadwiger-Finsler Inequality

$a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2 \geq 4\sqrt{3}F$.

Indeed, $(ab+bc+ca) - \left(a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2\right) =$
$a^2+b^2+c^2-ab-ac-bc \geq 0$.